

## Solve a system with non-linear equations

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$$\text{Given: } \begin{cases} x + y + z = 3 & \dots (1) \\ x^2y + y^2z + z^2x = 4 & \dots (2) \\ x^2 + y^2 + z^2 = 5 & \dots (3) \end{cases}$$

Solve for  $(x, y, z) \in \mathbf{R}$ .

### Preliminary notes:

**(a)** The expression  $x^2y + y^2z + z^2x$  in eq. (2) is cyclic but not symmetric.

A function  $f(x, y, z)$  is **cyclic** if  $f(x, y, z) = f(y, z, x) = f(z, x, y)$ .

A function is **symmetric** if the function is unaltered if we interchange any two of the given variables. e.g.  $x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2$  is symmetric.

The system is easier to solve if we have symmetric functions in all equations.

**(b)** Observe that the system is easier to solve if we need only integral solutions, i.e.  $(x, y, z) \in \mathbf{Z}$ .

Shrewd readers may observe (not prove) that  $(x, y, z) = (2, 1, 0)$ ,  $(0, 2, 1)$  and  $(1, 0, 2)$  are solutions. Please note that  $(0, 1, 2)$ ,  $(2, 0, 1)$  and  $(1, 2, 0)$  are not solutions because eq. (2) is symmetric and not cyclic.

### (A) Some traditional methods

They are a bit long, but very worthwhile to learn.

$$\text{From (1), } x = 3 - y - z \quad \dots (4)$$

$$\begin{aligned} (4) \downarrow (2), \quad & (3 - y - z)^2y + y^2z + z^2(3 - y - z) = 4 \\ & (9 + y^2 + z^2 - 6y - 6z + 2yz)y + y^2z + z^2(3 - y - z) = 4 \\ & y^3 - 6y^2 - 6yz + 9y - z^3 + 3z^2 + 3zy^2 = 4 \\ & y^3 + (3z - 6)y^2 + (9 - 6z)y - z^3 + 3z^2 - 4 = 0 \quad \dots (5) \end{aligned}$$

$$\begin{aligned} (4) \downarrow (3), \quad & (3 - y - z)^2 + y^2 + z^2 = 5 \\ & (9 + y^2 + z^2 - 6y - 6z + 2yz) + y^2 + z^2 = 5 \\ & 2y^2 + 2z^2 - 6y - 6z + 2yz + 4 = 0 \\ & y^2 + z^2 - 3y - 3z + yz + 2 = 0 \\ & y^2 + (z - 3)y + z^2 - 3z + 2 = 0 \quad \dots (6) \end{aligned}$$

Our attention is to solve a cubic equation (5) together with a quadratic equation (6).

#### (1) J. J. Sylvester (1814 – 1897) method

$$\begin{aligned} (5) \times y, \quad & y^4 + (3z - 6)y^3 + (9 - 6z)y^2 + (-z^3 + 3z^2 - 4)y = 0 \\ (5) \times 1, \quad & y^3 + (3z - 6)y^2 + (9 - 6z)y + (-z^3 + 3z^2 - 4) = 0 \\ (6) \times y^2, \quad & y^4 + (z - 3)y^3 + (z^2 - 3z + 2)y^2 = 0 \\ (6) \times y, \quad & y^3 + (z - 3)y^2 + (z^2 - 3z + 2)y = 0 \\ (6) \times 1, \quad & y^2 + (z - 3)y + (z^2 - 3z + 2) = 0 \end{aligned}$$

In order this set of equations to have solutions, the coefficient determinant must be zero.

$$D = \begin{vmatrix} 1 & 3z-6 & 9-6z & -z^3+3z^2-4 & 0 \\ 0 & 1 & 3z-6 & 9-6z & -z^3+3z^2-4 \\ 1 & z-3 & z^2-3z+2 & 0 & 0 \\ 0 & 1 & z-3 & z^2-3z+2 & 0 \\ 0 & 0 & 1 & z-3 & z^2-3z+2 \end{vmatrix} = 0$$

$$D = \begin{vmatrix} 1 & 3z-6 & 9-6z & -z^3+3z^2-4 & 0 \\ 0 & 1 & 3z-6 & 9-6z & -z^3+3z^2-4 \\ 0 & -2z+3 & z^2+3z-7 & z^3-3z^2+4 & 0 \\ 0 & 1 & z-3 & z^2-3z+2 & 0 \\ 0 & 0 & 1 & z-3 & z^2-3z+2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3z-6 & 9-6z & -z^3+3z^2-4 \\ -2z+3 & z^2+3z-7 & z^3-3z^2+4 & 0 \\ 1 & z-3 & z^2-3z+2 & 0 \\ 0 & 1 & z-3 & z^2-3z+2 \end{vmatrix}$$

$$= \begin{vmatrix} z^2+3z-7 & z^3-3z^2+4 & 0 \\ z-3 & z^2-3z+2 & 0 \\ 1 & z-3 & z^2-3z+2 \end{vmatrix} - (-2z+3) \begin{vmatrix} 3z-6 & 9-6z & -z^3+3z^2-4 \\ z-3 & z^2-3z+2 & 0 \\ 1 & z-3 & z^2-3z+2 \end{vmatrix}$$

$$+ \begin{vmatrix} 3z-6 & 9-6z & -z^3+3z^2-4 \\ z^2+3z-7 & z^3-3z^2+4 & 0 \\ 1 & z-3 & z^2-3z+2 \end{vmatrix}$$

$$= (6z^5 - 41z^4 + 104z^3 - 117z^2 + 52z - 4)$$

$$- (-2z+3)(3z^5 - 15z^4 + 14z^3 + 27z^2 - 39z + 2)$$

$$+ (3z^6 - 21z^5 + 85z^4 - 221z^3 + 303z^2 - 167z + 10)$$

$$= 9z^6 - 54z^5 + 117z^4 - 105z^3 + 27z^2 + 6z = 0$$

Therefore, after factorization,

$$3z(z-1)(z-2)(3z^3-9z^2+6z+1) = 0$$

$$z = 0 \text{ or } z = 1 \text{ or } z = 2$$

$$\text{or } z = \frac{2\sqrt{3}\cos\left(\frac{5\pi}{18}\right)}{3} + 1 \text{ or } z = \frac{2\sqrt{3}\cos\left(\frac{17\pi}{18}\right)}{3} + 1 \text{ or } z = \frac{2\sqrt{3}\cos\left(\frac{29\pi}{18}\right)}{3} + 1$$

Here we do not discuss various methods in solving cubic equation:  $3z^3 - 9z^2 + 6z + 1 = 0$ , we employ a trigonometric method. (Search the literature yourselves.)

For approximation roots we have:

$$z = 1.7422271989686, -0.1371580426033 \text{ or } 1.3949308436347$$

To find the corresponding values of  $x$  and  $y$ , you may use (6) and (4). It needs some work. It is better to observe that the given equations (1) - (3) are all cyclic, so the roots for  $y$  and also  $x$  are the same as those of  $z$ . Then your work is to see which  $y$  and  $z$  goes with which  $x$ . Direct substitutions may easily confirm your results.

Hence  $(x, y, z) = (2, 1, 0), (0, 2, 1), (1, 0, 2)$  Or  
 $(x, y, z) \approx (1.742, 1.395, -0.137), (-0.137, 1.742, 1.395), (1.395, -0.137, 1.742)$ .

## (2) Étienne Bézout (1730 – 1783) method

$$\text{Solving } y^3 + (3z - 6)y^2 + (9 - 6z)y + (-z^3 + 3z^2 - 4) = 0 \quad \dots (5)$$

$$y^2 + (z - 3)y + z^2 - 3z + 2 = 0 \quad \dots (6)$$

is not a easy task. A 5 x 5 determinant is not good to solve. The following gives only a 3 x 3 determinant.

$$\text{First we have : } \frac{y^3}{y^2} = \frac{(3z-6)y^2 + (9-6z)y + (-z^3+3z^2-4)}{(z-3)y + z^2 - 3z + 2}$$

$$(3z - 6)y^2 + (9 - 6z)y + (-z^3 + 3z^2 - 4) = (z - 3)y^2 + (z^2 - 3z + 2)y$$

$$\text{Simplify, } (2z - 3)y^2 + (-z^2 - 3z + 7)y + (-z^3 + 3z^2 - 4) = 0 \quad \dots (7)$$

$$\text{Also, } \frac{y+(3z-6)}{y+(z-3)} = \frac{(9-6z)y+(-z^3+3z^2-4)}{(z^2-3z+2)y}$$

$$(z^2 - 3z + 2)y^2 + (3z - 6)(z^2 - 3z + 2)y = (9 - 6z)y^2 + (9 - 6z)(z - 3)y$$

$$+ (-z^3 + 3z^2 - 4)y + (-z^3 + 3z^2 - 4)(z - 3)$$

$$(z^2 + 3z - 7)y^2 + (4z^3 - 12z^2 - 3z + 19)y + (z^4 - 6z^3 + 9z^2 + 4z - 12) = 0 \quad \dots (8)$$

In order (6), (7), (8) to have solutions,

$$\begin{vmatrix} 1 & z - 3 & z^2 - 3z + 2 \\ 2z - 3 & -z^2 - 3z + 7 & -z^3 + 3z^2 - 4 \\ z^2 + 3z - 7 & 4z^3 - 12z^2 - 3z + 19 & z^4 - 6z^3 + 9z^2 + 4z - 12 \end{vmatrix} = 0$$

$$9z^6 - 54z^5 + 117z^4 - 105z^3 + 27z^2 + 6z = 0$$

You can get the same equation as **J. J. Sylvester method**. The rest of the calculations are therefore left behind.

## (B) Change of variables

The **J. J. Sylvester method** and **Étienne Bézout method** show you the general ways to handle higher order system of equations. Due the cyclic property of the given set of equation, we can change variables to simplify our works.

$$(1) \text{ Let } x = u + 2, y = v + 1, z = w \quad \dots (9)$$

Then we change the original set of equations to:

$$u + v + w = 0 \quad \dots (10)$$

$$(u + 2)^2(v + 1) + (v + 1)^2w + w^2(u + 2) = 4 \quad \dots (11)$$

$$(u + 2)^2 + (v + 1)^2 + w^2 = 5 \quad \dots (12)$$

We use (9) because the constant terms of (10), (11) and (12) are zero, hoping to make life easier.

$$\text{From (10), } w = -(u + v) \quad \dots (13)$$

(13) ↓ (11),

$$(u + 2)^2(v + 1) - (v + 1)^2(u + v) + (u + v)^2(u + 2) = 4$$

$$(u^3 - v^3) + 3u^2 + 6uv + 3u + 3vu^2 + 3v = 0 \quad \dots (14)$$

(13) ↓ (12),

$$(u + 2)^2 + (v + 1)^2 + (u + v)^2 = 5$$

$$u^2 + uv + v^2 + 2u + v = 0 \quad \dots (15)$$

Note that we like to solve a new system of homogenous equations : (10), (14), (15).

From (15),

$$u^2 + uv + v^2 = -2u - v \quad \dots (16)$$

(16) ↓ (14),

$$(u - v)(-2u - v) + 3u^2 + 6uv + 3u + 3vu^2 + 3v = 0$$

$$u^2 + 7uv + 3u + v^2 + 3vu^2 + 3v = 0$$

$$(1 + 3v)u^2 + (7v + 3)u + (v^2 + 3v) = 0 \quad \dots (17)$$

(16) × (1 + 3v),

$$(1 + 3v)u^2 + (1 + 3v)(2 + v)u + (1 + 3v)(v^2 + v) = 0 \quad \dots (18)$$

(18) ↓ (16),

$$[(1 + 3v)(2 + v) - (7v + 3)]u + [(1 + 3v)(v^2 + v) - (v^2 + 3v)] = 0$$

$$[3v^2 - 1]u + 3v^3 + 3v^2 - 2v = 0$$

$$u = -\frac{3v^3 + 3v^2 - 2v}{3v^2 - 1} \quad \dots (19)$$

(19) ↓ (16),

$$\left(\frac{3v^3 + 3v^2 - 2v}{3v^2 - 1}\right)^2 - (2 + v)\left(\frac{3v^3 + 3v^2 - 2v}{3v^2 - 1}\right) + (v^2 + v) = 0$$

$$(3v^3 + 3v^2 - 2v)^2 - (2 + v)(3v^2 - 1)(3v^3 + 3v^2 - 2v) + (3v^2 - 1)^2(v^2 + v) = 0$$

$$9v^6 - 18v^4 + 3v^3 + 9v^2 - 3v = 0$$

$$3v(v - 1)(v + 1)(3v^3 - 3v + 1) = 0$$

$$v = 0 \text{ or } v = 1 \text{ or } v = -1 \text{ or } v = \frac{2\sqrt{3}\cos\left(\frac{5\pi}{18}\right)}{3} \text{ or } v = \frac{2\sqrt{3}\cos\left(\frac{17\pi}{18}\right)}{3} \text{ or } v = \frac{2\sqrt{3}\cos\left(\frac{29\pi}{18}\right)}{3}$$

Making use of (13) and (19) we can find the corresponding u and w and finally making use of (9) we can find all (x, y, z). Another better way is to find all values of y first and making use the cyclic property of the original set of equations, we can find all roots of (x, y, z). The checking steps are left to the readers.

(2) The following change of variables has more symmetry, but the system is not homogenous.

Let  $x = u + 1, y = v + 1, z = w + 1$ , system becomes

$$\begin{cases} (u + 1) + (v + 1) + (w + 1) = 3 \\ (u + 1)^2(v + 1) + (v + 1)^2(w + 1) + (w + 1)^2(u + 1) = 4 \\ (u + 1)^2 + (v + 1)^2 + (w + 1)^2 = 5 \end{cases}$$

$$\begin{cases} u + v + w = 0 \\ (u^2 + v^2 + w^2) + 2(uv + vw + wu) + (u^2v + v^2w + w^2u) + 3(u + v + w) = 1 \\ (u^2 + v^2 + w^2) + 2(u + v + w) = 2 \end{cases}$$

$$\begin{cases} u + v + w = 0 \\ (u + v + w)^2 + (u^2v + v^2w + w^2u) + 3(0) = 1 \\ (u^2 + v^2 + w^2) + 2(0) = 2 \end{cases}$$

$$\begin{cases} u + v + w = 0 & \dots (20) \\ u^2v + v^2w + w^2u = 1 & \dots (21) \\ u^2 + v^2 + w^2 = 2 & \dots (22) \end{cases}$$

From (20),  $w = -(u + v) \dots (23)$

(23) ↓ (21),  $u^2v - v^2(u + v) + (u + v)^2u = 1$   
 $(u^3 - v^3) + 3vu^2 = 1 \dots (24)$

(23) ↓ (22),  $u^2 + v^2 + (u + v)^2 = 2$   
 $u^2 + uv + v^2 = 1 \dots (25)$

(25) ↓ (24),  $(u - v) + 3vu^2 = 1$   
 $v(3u^2 - 1) = 1 - u$

Since  $u^2 \neq \frac{1}{3}, v = \frac{1-u}{3u^2-1} \dots (26)$

(26) ↓ (25),  $u^2 + u\left(\frac{1-u}{3u^2-1}\right) + \left(\frac{1-u}{3u^2-1}\right)^2 = 1$

$$(u^2 - 1)(3u^2 - 1)^2 - u(u - 1)(3u^2 - 1) + (u - 1)^2 = 0$$

$$(u - 1)[(u + 1)(3u^2 - 1)^2 - u(3u^2 - 1) + (u - 1)] = 0$$

$$(u - 1)[9u^5 + 9u^4 - 9u^3 - 6u^2 + 3u] = 0$$

$$(u - 1)[3u(u + 1)(3u^3 - 3u + 1)] = 0$$

$$u(u - 1)(u + 1)(3u^3 - 3u + 1) = 0$$

The rest is to solve:

$$\begin{cases} u(u - 1)(u + 1)(3u^3 - 3u + 1) = 0 \\ v = \frac{1-u}{3u^2-1} \\ w = -(u + v) \end{cases}$$

The solutions are the same as previous results and are omitted here.